Yiddish word of the day

" farlibn zikh " = פֿאַרליבן זיך

to fall in love !! =

Yiddish expression

" Di kinder in dayne yorn zoln beser makhn " = די קינדער אין דײַנע יאָרן זאָלן בעסער מאַכן

" may your children's lives be even better " =

6.3 - Matrices of Linear Transf.

Take teacher Evals !!!

Canvas (Due 8/24!)

- SETS (What do you think)
  - fill out for me (and Sam if you enrolled in section)

- if > 85% fill out evals everyone gets 5% boost on final exam.

!!!!

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$ we got the matrix A_T whose columns are $T(e_i)$

$$\Rightarrow T(\vec{v}) = A_T \vec{v}$$

Now: V, W with basis $\mathcal{B}_V = (v_1 \dots v_n)$
$\mathcal{B}_W = (w_1, \dots, w_m)$

- $T: V \to W$ then $T(v) = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m$
  - in particular $T(v_i) = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$
  - So can form $[T(v_i)]_{\mathcal{B}_W} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ in $\mathbb{R}^m$

We define the matrix of T with respect to $\mathcal{B}_V, \mathcal{B}_W$ as

$$A_{T,\mathcal{B}_V,\mathcal{B}_W} = \begin{pmatrix} [T(v_1)]_{\mathcal{B}_W} & [T(v_2)]_{\mathcal{B}_W} & & [T(v_n)]_{\mathcal{B}_W} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}_{m\times n}$$

$$\Longrightarrow \text{gives us} \quad \underbrace{[T(v)]_{\mathcal{B}_W}}_{\text{in } \mathbb{R}^m} = \underbrace{A_{T,\mathcal{B}_V,\mathcal{B}_W}}_{m\times n} \underbrace{[v]_{\mathcal{B}_V}}_{\text{in } \mathbb{R}^n}$$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B}_V = (1, x, x^2)$

$W = M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B}_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$T: V \to W$ as

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} 2a_0 & a_1 - a_2 \\ a_0 & 4a_1 \end{pmatrix}$$

Find $A_{T, \mathcal{B}_V, \mathcal{B}_W}$:

- $T(1) = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow [T(1)]_{\mathcal{B}_W} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- $T(x) = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} \Rightarrow [T(x)]_{\mathcal{B}_W} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix}$

- $T(x^2) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow [T(x^2)]_{\mathcal{B}_W} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$$A_{T, \mathcal{B}_V, \mathcal{B}_W} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}_{4\times 3}$$

Check: Consider the vector $v = 2 + x - x^2$ in V

- $T(v) = \begin{pmatrix} 2(2) & 1-(-1) \\ 2 & 4(1) \end{pmatrix} \Rightarrow [T(v)]_{\mathcal{B}_W} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$

- $[v]_{\mathcal{B}_V} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}}_{A_{T,\mathcal{B}_V,\mathcal{B}_W}} \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}_{[v]_{\mathcal{B}_V}} = 2\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} - 1\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

:)

$= \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix} = [T(v)]_{\mathcal{B}_W}$

Again let V, W vector spaces, $\mathcal{B}_V, \mathcal{B}_W$ bases.

$T: V \to W$	$A_{T,\mathcal{B}_V,\mathcal{B}_W} := A$ ($m \times n$ matrix)
$\ker(T)$	$\text{null}(A)$
$\text{nullity}(T)$	$\text{nullity}(A)$
$\text{Range}(T)$	$\text{col}(A)$
$\text{rank}(T)$	$\text{rank}(A)$
T is <u>injective</u>	columns of A LI
T is <u>surjective</u>	column space spans $\mathbb{R}^m$ (pivot in every row)
T is <u>isomorphism</u>	A is invertible (square)

ex) $T: \underset{V}{\underset{\parallel}{\mathbb{R}_2[x]}} \to \underset{W}{\underset{\parallel}{M_{2\times 2}(\mathbb{R})}}$ with $\mathcal{B}_V = (1, x, x^2)$

$\mathcal{B}_W: \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} 2a_0 & a_1 - a_2 \\ a_0 & 4a_1 \end{pmatrix}$$

Find $\ker(T)$

Use above connection, we'll find

$$\Rightarrow A_T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \xrightarrow[\text{form}]{\text{put into echelon}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Pivot in every column $\Rightarrow null(A_{T,\mathcal{B}_V,\mathcal{B}_W}) = \{0\} \Rightarrow \ker T = \{0\}$

ex2) $V = \mathbb{R}_3[x]$ $\quad \mathcal{B}_V = (1, x, x^2, x^3)$

$W = \mathbb{R}_2[x]$ $\quad \mathcal{B}_W = (1, x, x^2)$

$D: V \to W$ derivative $\quad (D(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$= a_1 + 2a_2 x + 3a_3 x^2$

Find A_{D,B_V,B_W} and find $\ker(D)$

- $D(1) = 0 \Rightarrow [D(1)]_{B_W} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $D(x) = 1$
- $D(x^2) = 2x$
- $D(x^3) = 3x^2$

$$\Rightarrow A_{D,B_V,B_W} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow \text{null}(A_{D,B_V,B_W}) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = [v]_{B_V} \Rightarrow v = 1(1) + 0(x) + 0(x^2) + 0(x^3)$$

$$v = 1$$

$\ker(D) = \text{span}(v) = \text{span}(1) = \text{constants}$

ex) $V = M_{2\times 2}(\mathbb{R})$ $W = \mathbb{R}^4$

$B_V = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ $B_W = (e_1, e_2, e_3, e_4)$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \\ a+c \\ d \end{pmatrix}$$

Find A_{T,B_V,B_W}

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]_{B_W} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right]_{B_W} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}_W}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right]_{\mathcal{B}_W}$$

$$\Rightarrow A_{T,\mathcal{B}_V,\mathcal{B}_W} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q: Is T injective? If not find $\ker T$

ex) Change of basis matrix

V vector space with two bases $\mathcal{B}_1$, $\mathcal{B}_2$

$\mathcal{B}_1 = (v_1 \ldots v_n)$, $\mathcal{B}_2 = (w_1 \ldots w_n)$

Then consider the identity transformation

$$I : (V, \mathcal{B}_1) \longrightarrow (V, \mathcal{B}_2) \qquad I(v) = v$$

$$A_{I,\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix} [v_1]_{\mathcal{B}_2} & [v_2]_{\mathcal{B}_2} & \cdots & [v_n]_{\mathcal{B}_2} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$= P_{\mathcal{B}_1 \to \mathcal{B}_2}$$

$$\left([v]_{\mathcal{B}_2} = P_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_1} \right)$$

$$[T(v)]_{\mathcal{B}_2} = A_{T,\mathcal{B}_1,\mathcal{B}_2} [v]_{\mathcal{B}_1}$$

Determinants for Transformations

We want to apply the concept of determinants to

$T: V \to V$ (linear operator)
$\mathcal{B}$ $\mathcal{B}$

We just showed how to get a matrix $A_{T,\mathcal{B},\mathcal{B}}$ from T

Can we define $\det(T) = \det(A_{T,\mathcal{B},\mathcal{B}})$?

- A problem is, does this definition depend on choice of basis

• If $\mathcal{B}'$ is another basis for V, we need to know the relation between

$$A_{T,\mathcal{B}',\mathcal{B}'} \quad \text{and} \quad A_{T,\mathcal{B},\mathcal{B}}$$

turns out

$$\Rightarrow A_{T,\mathcal{B}',\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}\, A_{T,\mathcal{B},\mathcal{B}}\, P_{\mathcal{B}'\to\mathcal{B}}$$

$$\boxed{A_{T,\mathcal{B}',\mathcal{B}'} = \left(P_{\mathcal{B}'\to\mathcal{B}}\right)^{-1} A_{T,\mathcal{B},\mathcal{B}}\, P_{\mathcal{B}'\to\mathcal{B}}}$$

$$\Rightarrow \det\left(A_{T,\mathcal{B}',\mathcal{B}'}\right) = \det\left(\left(P_{\mathcal{B}'\to\mathcal{B}}\right)^{-1} A_{T,\mathcal{B},\mathcal{B}}\, P_{\mathcal{B}'\to\mathcal{B}}\right)$$

$$= \det\left(\left(P_{B' \to B}\right)^{-1}\right) \det\left(A_{T,B,B}\right) \det\left(P_{B' \to B}\right)$$

$$= \det\left(A_{T,B,B}\right) \det\left(P_{B' \to B}^{-1}\right) \det\left(P_{B' \to B}\right)$$

$$= \det\left(A_{T,B,B}\right) \det\left(P_{B' \to B}^{-1} P_{B' \to B}\right)$$

$$= \det\left(A_{T,B,B}\right)$$

So $\boxed{\det\left(A_{T,B',B'}\right) = \det\left(A_{T,B,B}\right)}$!

So we define $\det(T) := \det\left(A_{T,B,B}\right)$

STOP HERE!!

Chapter 7 - Eigenvalues / vectors (For Friday's Class!)

- We saw that we get a diff matrix for $T: V \to V$ for each basis.
  - Is there a "nice" basis that makes the matrix of T as simple as possible.

    ex) For example, is there a basis for V such that the matrix of T with respect to this basis is diagonal.

$$A_{T,\mathcal{B}} = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$$

- If $\mathcal{B} = (v_1 \ldots v_n)$ is a basis where the matrix of T is as above then $T(v_1) = a_1 v_1$, $T(v_2) = a_2 v_2$, ..., $T(v_n) = a_n v_n$

Def: $T: V \to V$, and $0 \neq v$ a vector in V.

Then if $T(v) = \lambda v$ for some λ, we call v an eigenvector

and eigenvalue λ

ii) A $n \times n$ matrix. A nonzero vector $\vec{0} \neq \vec{v}$ in $\mathbb{R}^n$ is eigenvector of A if $A\vec{v} = \lambda\vec{v}$ (this λ is called eigenvalue.)

How to find eigenvectors/values for a given matrix:

- Suppose $\vec{v}$ is an eigenvector for A, so $A\vec{v} = \lambda\vec{v}$

$\Rightarrow$ $A\vec{v} - \lambda\vec{v} = \vec{0}$ $\quad \left(I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$

$(A - \lambda I_n)\vec{v} = \vec{0}$

$\Rightarrow$ $0 \neq \vec{v}$ is in null$(A - \lambda I_n)$

$\Rightarrow$ this matrix is not invertible.

$\Rightarrow$ $\det(A - \lambda I_n) = 0$!

$$\boxed{\begin{array}{l}\bullet \text{ So we set } \det(A - \lambda I_n) = 0 \text{ and solve for } \lambda \\ \text{Then we find the null}(A - \lambda I)\end{array}}$$

ex) $A = \begin{pmatrix} 5 & 4 \\ -2 & -1 \end{pmatrix}$ Find eigenvalues / vectors of A.

1) $A - \lambda I_2 = \begin{pmatrix} 5-\lambda & 4 \\ -2 & -1-\lambda \end{pmatrix}$

2) $\det(A - \lambda I_2) = (5-\lambda)(-1-\lambda) - (4)(-2)$

$= -5 - 5\lambda + \lambda + \lambda^2 + 8$

$= \lambda^2 - 4\lambda + 3$

3) Solve for lambda:

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0$

$\lambda_1 = 1$: null $\begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{R_1/4} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\boxed{\text{null space} = \text{Span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

So $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\underline{1}$

$\underline{\text{Check}}$: $A\vec{v}_1 = \begin{pmatrix} 5 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 5 \\ -2 \end{pmatrix} + 1\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 3}$: $\begin{pmatrix} 2 & 4 \\ -2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ null space = span$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\underline{3}$

$\underline{\text{Check}}$: $A\vec{v}_2 = \begin{pmatrix} 5 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 5 \\ -2 \end{pmatrix} + 1\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -2 \\ 1 \end{pmatrix}$